Exact Nonlinear Control of Spacecraft Slewing Maneuvers with Internal Momentum Transfer

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The recent literature on spacecraft slewing maneuvers relies on two techniques for taking nonlinear effects into acount: suboptimal torque commands are either generated by continuation techniques from open-loop laws arising from Pontryagin's Maximum Principle, or obtained by polynomial truncation of analytic feedback laws arising from Bellman's Optimality Principle. Correction for suboptimality and unmodeled disturbances must then be carried out on-line based on a time-varying approximation of the error dynamics along the nominal state trajectory. In this paper it is shown how exact command generation and tracking can be obtained by standard linear methods through the prior construction of nonlinear coordinate transformations together with nonlinear feedback. The nonlinear equations of motion of a spacecraft controlled by internal reaction wheels are thereby transformed into three decoupled double integrators, driven by acceleration commands in attitude parameter space. Maneuver specifications are then transformed into the equivalent linear representation thus obtained, and codified into linear optimal control or path-planning problems with exact solutions, which are then transformed back. Moreover, correction for disturbances can also be carried out in the transformed linear representation, making gain scheduling unnecessary. Hardware and software implementation of the linearizing transformations is discussed, and corroborating simulation results are presented.

Introduction

CONTROL laws and control systems for sequential single-axis spacecraft rotational maneuvers can be designed effectively with linear techniques, such as by Breakwell[1] and Hefner et al.[2] and with new techniques by Hefner,[3] Turner and Chun,[4] and Juang et al.[5] The nonlinearity of the equations of rotational motion must be taken into account, however, to satisfy the increasingly stringent requirements of agile spacecraft undergoing fast multiaxial slews of large amplitude.

Such maneuvers are customarily formulated as nonlinear optimal control problems, and then approximately solved by essentially one of two approaches: either a suboptimal command is generated by relaxation or continuation methods from the coupled state and costate equations derived from Pontryagin's Maximum Principle, such as by Junkins and Turner[6] as well as Vadali and Junkins,[7,8] or else by polynomial truncations of optimal analytic feedback laws derived from Bellman's Optimality Principle, as done by Dwyer[9] and Dwyer and Sena[10] as well as by Carrington and Junkins.[11,12]

Such techniques have been found to be sufficiently accurate with few iterations for sufficiently slow maneuvers. However, implementable exact solutions, in the sense of guaranteeing zero terminal error with bounded controls in the absence of disturbances, have not been obtained by such means, except for the case of pure rigid detumbling with variable external torque controls.[13]

Moreover, even when a sufficiently accurate nominal solution is obtained, the problem of correcting for disturbances during the maneuver leads to time-varying linear quadratic Gaussian (LQG) problems that have to be solved along the nominal trajectory, hence, at best, approximately implemented by gain scheduling.

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In contrast, the program begun by Dwyer[14] for the case of external torque actuators, and continued here with internal reaction wheel actuators, provides *exact* nominal solutions, practically implementable in software, or with hardware summers, scalers, and integrators, by a proper choice of attitude and rate variables, followed by the application of two analytical artifices: appropriate invertible nonlinear coordinate transformations coupled with nonlinear feedback are first found for each system (not each maneuver) so that an equivalent linear model of the system equations is obtained, as advocated by Hunt et al.[15] The *transformed* maneuver specifications are then formulated as linear optimal control problems, whence torque commands are obtained by means of the inverse transformations.

In addition to providing an exact nominal solution, often in closed form, this approach also permits the design of a *single* linear regulator for the correction of disturbances, producing incremental corrections to the *transformed* command in response to the *transformed* error states. The full power of recently developed methods[3-5] for linear regulation and tracking can thus be applied to general nonlinear maneuvers. The application of such linear methods to vibration and noise suppression during nonlinear slews will be reported in greater detail elsewhere, based upon the nominal command generation algorithms developed in Ref. 14 and in the present paper. Appended simulation results illustrate the effectiveness of the proposed command generation method in comparison with direct suboptimal nonlinear control.

Spacecraft and Reaction Wheel Dynamics

The spacecraft model considered by Vadali and Junkins[7] will be employed here. The model consists of a rigid main body equipped with three reaction wheels mounted coaxially with the yaw, pitch, and roll axes originating from the spacecraft's center of mass, as shown in Fig. 1.

Some notation is in order: I^0 will denote the system inertia matrix (main body principal moments, transverse wheel moments, and axial wheel moments), while I^A will represent the diagonal matrix of axial wheel moments alone. By ω will be meant the column matrix of components of inertial angular velocity of the spacecraft's main body resolved along the yaw, pitch, and roll axes, while Ω will denote the column matrix of axial angular velocities of each reaction wheel. Finally, h will stand for the column matrix of components of the total angular momentum of the system, likewise resolved along the principal spacecraft axes.

As is shown in Ref. 7, the total spacecraft angular momentum is given by

$$h = I^0\omega + I^A\Omega \tag{1}$$

in terms of the spacecraft and wheel inertias and angular velocities. (A more general expression is given in Refs. 8 and 12, which permits more than three reaction wheels. However, the three-wheel example will be retained here for simplicity.)

In the absence of external torques, conservation of angular momentum applied to Eq. (1) yields

$$I^0\dot{\omega} + I^A\dot{\Omega} + \omega \times h = 0 \tag{2}$$

where a dot denotes a time derivative and $\times$ a cross product, the latter arising because of the rotation of the body-fixed reference frame. [More generally, the right-hand side of Eq. (2) stands for the net external torque with respect to the spacecraft center of mass, also resolved along body axes.]

The rate of change of angular momentum about the center of mass of each reaction wheel likewise equals the corresponding wheel motor torques τ_i, which are shown in Ref. 7 to be jointly given by

$$I^A(\dot{\Omega} + \dot{\omega}) = \tau \tag{3}$$

There is no cross-product term because each component of Eq. (3) corresponds to a pure spin. (Again a more general expression holds for other wheel configurations, as is given in Refs. 7 and 12.)

Insertion of the expression for $I^A\dot{\Omega}$ obtained from Eq. (3) into Eq. (2) finally yields the evolution equation for ω in terms of the wheel torques τ and the total angular momentum h:

$$[I^0 - I^A]\dot{\omega} = h \times \omega - \tau \tag{4}$$

In general, it is necessary to propagate the total angular momentum h (or, equivalently, the wheel angular velocities Ω hidden therein) to complete the dynamic model. However, in the absence of external torques it is possible, as in Ref. 7, to represent the angular momentum h in terms of the column matrix h^I of its (necessarily constant) *inertial* components, through the time-varying matrix C that transforms the underlying inertial reference frame into the principal spacecraft axes, as will be shown next.

Kinematic Representation of the Angular Momentum

If the instantaneous spacecraft orientation is represented by a virtual rotation of ϕ rad about a unit vector e, then the change of basis matrix C from inertial to body reference can, as usual, be parameterized by the symmetric Euler quaternion

$$\beta = \mathrm{col}(\beta_0, \beta') \tag{5}$$

where $\beta_0 = \cos(\frac{1}{2}\phi)$ and $\beta' = \sin(\frac{1}{2}\phi)e$, as in Ref. 6 ("col" denotes a column matrix). The change of basis transformation is then given by

$$C = C(\beta) = [2\beta'\beta'^T + (\beta_0^2 - \beta'^T\beta')I - 2\beta_0\beta' \times] \tag{6}$$

where I is the 3×3 identity matrix, T denotes transposition, and $\beta' \times$ is again the matrix representation of the cross product with β'.

The inertial angular momentum h^I can then be measured at the start of a maneuver by

$$h^I = C(\beta(0))^{-1}\{I^0\omega(0) + I^A\Omega(0)\} \tag{7}$$

where the components of the inertial angular velocity $\omega(0)$ are measured by yaw, pitch, and roll gyros, while the initial angular velocities $\Omega_i(0)$ of the wheels can be given by tachometers coaxially mounted with the wheel motors, with $\beta(0)$ encoding the initial orientation determined by the attitude determination system. Once h^I is determined, the angular momentum in body-fixed coordinates is given by

$$h = h(\beta(t)) = C(\beta(t))h^I \tag{8}$$

(A simplified representation is used in Ref. 7 that requires the preselection of a preferential inertial reference frame. However, it will be seen below that in the approach being advocated here the choice of inertial reference frame is determined instead by maneuver-dependent requirements.)

To complete the equations of motion the evolution of the kinematic parameter β must be taken into account, as is discussed next.

Spacecraft Attitude Kinematics

The evolution of the attitude quaternion can be represented in terms of quaternion algebra, as shown in Ref. 10, but herein is decomposed into the evolution of its scalar part β_0 and its vector part β' to yield, as in Ref. 14,

$$\dot{\beta}_0 = -\tfrac{1}{2}\omega^T\beta' \tag{9a}$$

$$\dot{\beta}' = \tfrac{1}{2}(\beta_0\omega + \beta' \times \omega) \tag{9b}$$

In view of the identity

$$\beta_0^2 + \beta'^T\beta' = 1 \tag{10}$$

that follows from the definition of β in Eq. (5), the scalar part [Eq. (9a)] of the kinematic equations may be omitted, provided the spacecraft maneuver (or the choice of inertial frame of reference) is constrained to avoid having $\beta_0 = 0$ (or equivalently $\beta'^T\beta' = 1$); that is, provided that the instantaneous spacecraft orientation is always represented by a virtual rotation such that $-\pi < \phi < \pi$. One may then let

$$\begin{aligned} \gamma &= \beta' \quad \text{and} \quad \gamma_0 = \beta_0 \quad \text{if} \quad \beta_0 > 0 \\ &= -\beta' \text{and} \quad\;\; = -\beta_0 \quad \text{if} \quad \beta_0 < 0 \end{aligned}$$

whence it follows that

$$\gamma_0 = \sqrt{1 - \gamma^T\gamma} \tag{11}$$

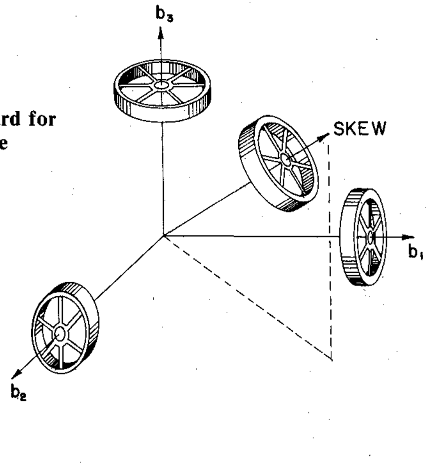

Fig. 1 NASA standard for reaction wheel attitude control system.

Any orientation is equally indexed by β or $-\beta$ (cf. Ref. 14) so that in effect $\gamma=\beta'$ for that attitude quaternion for which $\beta_0>0$.

Reduced Equations of Motion

Given that

$$C(\beta)^{-1}=C(\beta^*) \tag{12}$$

where $\beta^*=\mathrm{col}(\beta_0,-\beta')$ is the inverse of β in quaternion algebra (cf. Ref. 10), the total angular momentum in inertial coordinates, h^I, can be measured in terms of the initial reduced attitude parameter $\gamma(0)$ by the formula

$$h^I=C(-\gamma(0))\{I^0\omega(0)+I^A\Omega(0)\} \tag{13}$$

where the change of bases is rewritten as

$$C=C(\gamma)=[2\gamma\gamma^T+(\gamma_0^2-\gamma^T\gamma)I-2\gamma_0\gamma\times] \tag{14}$$

The angular momentum in body-fixed coordinates h is then also given by

$$h=h(\gamma(t))=C(\gamma(t))h^I \tag{15}$$

The reduced equations of motion are then found from Eqs. (9b) and (4) to be

$$\dot{\gamma}=\Gamma(\gamma)\omega \tag{16a}$$

$$J\dot{\omega}=h(\gamma)\times\omega-\tau \tag{16b}$$

where

$$J=I^0-I^A \quad \text{and} \quad \Gamma(\gamma)=\tfrac{1}{2}[\gamma_0 I+\gamma\times] \tag{17}$$

Nonlinear Optimal Control Formulation

The customary way of formulating an optimal maneuver, say, for the terminal reorientation to coincide with the inertial axes, is to pose the optimal control problem of minimizing the performance index

$$\tfrac{1}{2}p_1\beta'(t_f)^T\beta'(t_f)+\tfrac{1}{2}p_2\omega(t_f)^T\omega(t_f)$$
$$+\tfrac{1}{2}\int_0^{t_f}\{q_1\beta'(t)^T\beta'(t)+q_2\omega(t)^T\omega(t)+r\tau(t)^T\tau(t)\}\mathrm{d}t \tag{18}$$

subject to the dynamic and kinematic equations (4) and (9). Zero terminal error in finite time can be obtained only when it is possible to let $p_1,p_2\to\infty$.

In Refs. 7 and 8 an open-loop approach is followed, with the state equations (4) and (9) and corresponding costate equations iteratively solved as a two-point boundary value problem by the relaxation or continuation method.

In the approach taken by Carrington and Junkins,[12] a closed-loop point of view is taken, with the optimal nonlinear feedback law represented by a linear term followed by progressively more nonlinear corrections. This can be done recursively by linear processes, as shown by Dwyer in Refs. 10 and 16 and demonstrtated numerically in Ref. 12.

In all of these instances, only approximations to the optimal torque profile can be obtained, since solutions cannot be given in closed form. (It should be noted that the full attitude quaternion is propagated in these references, but only its vector part β' is penalized, as was found to be particularly necessary to obtain convergence of the Riccati equation that generates the linear feedback term.)

Equivalent Linear Formulation

The approach followed in Ref. 14 as in the present paper exploits the fact that the *reduced* equations of motion [Eqs. (16)] can be *exactly transformed* into an *equivalent linear system* (in fact, into three decoupled double integrators) by appropriate change of coordinates together with appropriate nonlinear feedback; the new state vector is defined by

$$x=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\gamma\\\dot{\gamma}\end{bmatrix}=\begin{bmatrix}\gamma\\\Gamma(\gamma)\omega\end{bmatrix} \tag{19}$$

[cf. Eqs. (16) and (17)], and the new input is given by

$$u=\ddot{\gamma}=\Gamma(\gamma)J^{-1}\{h(\gamma)\times\omega-\tau\}-\tfrac{1}{4}\omega^T\omega\gamma \tag{20}$$

as is shown in Appendix A. The transformed equations of motion thus become

$$\dot{x}_1=x_2 \tag{21a}$$

$$\dot{x}_2=u \tag{21b}$$

The attitude and angular rate variables can be recovered from the formulas

$$\gamma=x_1 \tag{22a}$$

$$\omega=2(x_0x_2+x_0^{-1}x_1x_1^Tx_2-x_1\times x_2) \tag{22b}$$

where

$$x_0=\sqrt{1-x_1^Tx_1} \tag{23}$$

as is also shown in Appendix A. The torque τ can then be recovered from the transformed input u according to the fundamental inversion formula

$$\tau=-2J[\gamma_0 I+\gamma_0^{-1}\gamma\gamma^T-\gamma\times]u$$
$$-\tfrac{1}{2}\gamma_0^{-1}\omega^T\omega J\gamma+h(\gamma)\times\omega \tag{24}$$

likewise derived in Appendix A.

Given the transformations [Eqs. (19-24)] various maneuvers can be performed *exactly*, among which model (path) following and terminal control will be discussed below.

Path Following

By path following is meant following a desired attitude trajectory $\gamma^*(t)$. The optimal torque command is then found by substitution of the nominal attitudes, rates, and accelerations

$$\gamma=\gamma^* \tag{25a}$$

$$\omega=2\{\gamma_0^*\dot{\gamma}^*+(\gamma_0^*)^{-1}\gamma^*\gamma^{*T}\dot{\gamma}^*-\gamma^*\times\dot{\gamma}^*\} \tag{25b}$$

$$u=\ddot{\gamma}^* \tag{25c}$$

into Eq. (24), with $\gamma(t_0)=\gamma^*(t_0)$.

Care must be exercised in choosing the attitude trajectory $\gamma^*(t)$ so that $\gamma_0^*(t)>0$ throughout, otherwise the torque τ becomes unbounded. Since usually only the initial and terminal attitudes or possible discrete intermediate attitudes are specified, $\gamma^*(t)$ can be shaped accordingly. This is similar to "path planning" for robot manipulators, cf., Ref. 17.

Terminal Control

By terminal control is meant determining a bounded command profile for reaching a desired terminal orientation. For an infinite horizon one is reduced to a regulator problem. In these cases the advantage of linearity in the transformed state equations (21) is lost if the transformed performance index is not quadratic. Therefore the angular velocity and torque penalties in the performance index [Eq. (18)] are replaced here by penalties on the parametric velocity $\dot{\gamma}$ and acceleration $\ddot{\gamma}$, i.e., Eq. (18) is replaced by

$$\tfrac{1}{2}p_1\gamma(t_f)^T\gamma(t_f)+\tfrac{1}{2}p_2\dot{\gamma}(t_f)^T\dot{\gamma}(t_f)+\tfrac{1}{2}\int_0^{t_f}\{q_1\gamma(t)^T\gamma(t)$$

$$+q_2\dot{\gamma}(t)^T\dot{\gamma}(t)+r\ddot{\gamma}(t)^T\ddot{\gamma}(t)^T\}\mathrm{d}t \tag{26}$$

which is equivalent to

$$\tfrac{1}{2}\boldsymbol{x}(t_f)^T\boldsymbol{P}\boldsymbol{x}(t_f)+\tfrac{1}{2}\int_0^{t_f}\{\boldsymbol{x}(t)^T\boldsymbol{Q}\boldsymbol{x}(t)+\boldsymbol{u}(t)^T\boldsymbol{R}\boldsymbol{u}(t)\}\mathrm{d}t \tag{26*}$$

where

$$\boldsymbol{P}=\mathrm{diag}(p_1\boldsymbol{I},p_2\boldsymbol{I}),\quad \boldsymbol{Q}=\mathrm{diag}(q_1\boldsymbol{I},q_2\boldsymbol{I})$$

and

$$\boldsymbol{R}=r\boldsymbol{I}$$

with

$$p_1,p_2,q_1,q_2\geq 0$$

and

$$r>0$$

This latter performance index is to be minimized subject to the transformed state equations (21), with the weight q_1 chosen sufficiently large so that $x_1(t)^Tx_1(t)<1$ for $0\leq t\leq t_f$ (for positive $x_0=\gamma_0$), with q_2 allowed to be zero.

Another approach, followed in Ref. 14 for the case of attitude control with external torques, is to set $q_1=q_2=0$, whence solutions in closed form with $p_1,p_2\to\infty$ can be obtained, as shown in Appendix B, and then to determine the locus of initial conditions for which one has

$$\max_{0\leq t\leq t_f} x_1(t)^Tx_1(t)<1$$

This in turn reduces to evaluating $x_1(t)$ at (at most two) real roots of $(\mathrm{d}/\mathrm{d}t)\,x_1(t)^Tx_1(t)$ for $0<t<t_f$, as is discussed more fully in Appendix C. It turns out that $\gamma_0(t)$ remains positive with $q_1=q_2=0$ for maneuvers where the initial angular rates are not too large, and in particular for the important case of rest-to-rest maneuvers. The optimal transformed state and input trajectories obtained for $q_1=q_2=0$ and after letting $p_1,p_2\to\infty$ are shown in Appendix B to be given in terms of (t/t_f) by

$$x_1(t)=\gamma(t)=2\{1-(t/t_f)\}^2\{\tfrac{1}{2}+(t/t_f)\}\gamma(0)$$

$$+\{1-(t/t_f)\}^2(t/t_f)t_f\dot{\gamma}(0) \tag{27}$$

$$x_2(t)=\dot{\gamma}(t)=-(6/t_f)\{1-(t/t_f)\}(t/t_f)\gamma(0)$$

$$+3\{1-(t/t_f)\}\{\tfrac{1}{3}-(t/t_f)\}\dot{\gamma}(0) \tag{28}$$

$$u(t)=\ddot{\gamma}(t)=(12/t_f^2)\{(t/t_f)-\tfrac{1}{2}\}\gamma(0)$$

$$+(6/t_f)\{(t/t_f)-\tfrac{2}{3}\}\dot{\gamma}(0) \tag{29}$$

Command Generation

On-board command generation for exact terminal control can be carried out simply by exciting three integrators with step inputs with amplitudes given by $(6/t_f^2)\{(2/t_f)\gamma_i(0)+\dot{\gamma}_i(0)\}$ and with integrator initial conditions given by $-(2/t_f)\{(3/t_f)\gamma_i(0)+2\dot{\gamma}_i(0)\}$ for each channel, as follows from differentiating Eq. (29), which is seen to consist of ramp inputs.

The input voltages to the electric motors driving the reaction wheels can then be generated by passing the ramp signals $u_i(t)$ through a memoryless interface, with nonlinear characteristics given by the inversion formula [Eq. (24)] (under the presumption that the motor torques are proportional to their input voltages).

The nonlinear interface can be "hard wired" off line either with dedicated integrated circuits or with microprocessors, inasmuch as the transformations can be implemented with summers, scalers, and multipliers alone. Indeed, $\gamma_0=|\beta_0|$ can be synthesized from $\gamma=\pm\beta'$, but it can also be measured by the same attitude determination sensors and processors that detect the other Euler attitude parameters. Alternatively, the inversion formula [Eq. (24)] can be coded as a fast nonrecursive subroutine within the autopilot (command generation) software.

Correction for Disturbances

Disturbances are generally introduced in the hardware implementation, such as the effects of parameter variations, e.g., oscillations in value of inertia matrices due to flexible modes, or else distortions within the wheel motor dynamics, as well as sensor noise. Thus, in general, the actual attitude and rate variables will not coincide with their nominal values.

The latter can be generated *in the transformed coordinates* $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ simply by integrating the nominal command $\boldsymbol{u}$ once to obtain $\boldsymbol{x}_2$ and twice for $\boldsymbol{x}_1$. Those values can then be compared on-line with the *transformed* measured states via interfaces with nonlinear characteristics given by Eq. (19).

A *time-invariant linear regulator* can then be designed to produce a correction signal $\Delta\boldsymbol{u}$ to be added to the nominal command *before* insertion thereof into the interface implementing Eq. (24) to excite the wheel actuators.

The complete control system architecture is similar to the one designed in Ref. 14 for the case of control with external torque commands, except for the need to set the initial angular momentum $\boldsymbol{h}(\gamma(0))$ in the command generation loops.

Parameter variations affecting the interface between controller and actuators, such as the inertia matrix in Eq. (20), are the most difficult to handle directly by the present method, but can be corrected if accelerometer feedback is available, as discussed in the case of robotics in Ref. 18.

Finally, the possibility of actuator saturation can be accounted for by periodically resetting the command generation program with the saturated state values, as discussed in Ref. 19 in robotics and in Ref. 20 for spacecraft maneuvers; for this it is better to use a least-mean-square attitude parameter jerk (i.e., the third derivative of γ) to be able to specify zero initial and terminal accelerations, as in Ref. 21.

Discussion and Conclusions

It has been shown that the equations of motion for a spacecraft driven by internal reaction wheels can be globally linearized in three steps, as is reviewed below.

In step 1, the absence of a net external torque was shown to permit the elimination of the wheel dynamics, which can be expressed in terms of the Euler attitude parameters and the angular velocity of the body-fixed reference frame.

In step 2, the restriction of permissible maneuvers to those that avoid the zero value for the scalar part of the Euler attitude quaternion (with respect to the otherwise freely chosen inertial reference frame), permitted the reduction of the four-dimensional Euler attitude quaternion equations to three-dimensional "reduced" Euler parameter equations.

In the third and most important step, the reduced state equations were transformed into decoupled double integrators by redefining the new state to consist of the reduced attitude parameter and its first time derivative, and the new input to be the second time derivative.

Given the reduced, then globally linearized, state equations, command generation tasks such as planning and following a prescribed attitude trajectory, as well as terminal control with bounded accelerations, were then formulated as analytically solvable linear problems with state inequality constraints that can therefore be tested analytically.

The present slew control scheme is more directly implementable than a similar design proposed earlier for the case of external torque commands, inasmuch as reaction wheel torques can be varied precisely with electromechanical actuators, whereas the usual hydrazine jets use for external torque control are not easily throttled to follow other than bang-bang commands.

It has been seen that the reduced quaternion yields an inequality constraint in the transformed systems, which can still be handled by linear optimal control. If the full quaternion were used, and equality constraint would occur that cannot be handled by linear optimal control.

It should also be pointed out that if variable external torques are available then exact prior detumbling can be carried out by linear means, as has been shown by the author earlier.[13] In any case, bang-bang control with external torques can be used to bring the angular velocity down to levels that do not excite the feedback singularity, which can be mapped out off line as described in Appendix C of the present paper. One is then reduced to a rest-to-rest maneuver that can be performed exactly without ever exciting the feedback singularity: cf. Appendix D.

Appendix A: Derivation of the Linearizing Transformations

The cross-product operation in spacecraft coordinates is given by the matrix

$$w\times=\begin{bmatrix}0 & -w_3 & w_2\\ w_3 & 0 & -w_1\\ -w_2 & w_1 & 0\end{bmatrix}$$

and likewise for the symbol $\gamma\times$.

The following formulas will be needed:

$$\Gamma(\gamma)^{-1}=2[\gamma_0 I+\gamma_0^{-1}\gamma\gamma^T-\gamma\times] \tag{A1}$$

$$\frac{\partial}{\partial\omega}\Gamma(\gamma)\omega=\Gamma(\gamma)=½[\gamma_0 I+\gamma\times] \tag{A2}$$

$$\frac{\partial}{\partial\gamma}\gamma_0=\frac{\partial}{\partial\gamma}(1-\gamma^T\gamma)^{½}=-\gamma_0^{-1}\gamma^T \tag{A3}$$

$$\frac{\partial}{\partial\gamma}\Gamma(\gamma)\omega=\frac{\partial}{\partial\gamma}½(\omega\gamma_0+\gamma\times\omega)=-½[\gamma_0^{-1}\omega\gamma^T+\omega\times] \tag{A4}$$

$$\gamma^T\Gamma(\gamma)\omega=½\{\gamma_0\gamma^T\omega+\gamma^T(\gamma\times\omega)\}=½\gamma_0\gamma^T\omega \tag{A5}$$

$$\omega\times\Gamma(\gamma)\omega=½\omega\times(\gamma\times\omega)=½(\omega^T\omega\gamma-\omega\omega^T\gamma) \tag{A6}$$

$$\Gamma(\gamma)^{-1}\gamma=2(\gamma_0\gamma+\gamma_0^{-1}\gamma\gamma^T\gamma-\gamma\times\gamma)$$

$$=2\{\gamma_0\gamma+\gamma_0^{-1}(1-\gamma_0^2)\gamma\}=(2/\gamma_0)\gamma \tag{A7}$$

Equation (22b) for ω follow from Eqs. (19) and (A1).

Equation (20) for u follows from the insertion of Eqs. (A2) and (A4-A6), together with Eq. (16a) for $\dot\gamma$ and (16b) for $\dot\omega$, into the following chain rule formula:

$$u=\ddot\gamma=\frac{d}{dt}\dot\gamma=\left\{\frac{\partial}{\partial\gamma}\Gamma(\gamma)\omega\right\}\dot\gamma+\left\{\frac{\partial}{\partial\omega}\Gamma(\gamma)\omega\right\}\dot\omega \tag{A8}$$

Finally, Eq. (20) yields

$$\tau=-J\,\Gamma(\gamma)^{-1}(u+¼\omega^T\omega\gamma)+h(\gamma)\times\omega=-J\,\Gamma(\gamma)^{-1}u$$

$$-¼\omega^T\omega J\,\Gamma(\gamma)^{-1}\gamma+h(\gamma)\times\omega \tag{A9}$$

whence Eq. (24) for τ follows from the insertion above of Eq. (A7).

Appendix B: Derivation of the Optimal Control Solution

The minimizing control with $q_1=q_2=0$ but finite p_1 and p_2 is given by the standard feedback law

$$u(t)=-(1/r)[0\ I]P(t_f-t)x(t) \tag{B1}$$

where $P(\tau)$ is the positive definite solution of the matrix differential Riccati equation

$$P(\tau)=P(\tau)\begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\ I & 0\end{bmatrix}P(\tau)$$

$$-P(\tau)\begin{bmatrix}0\\ I\end{bmatrix}(1/r)[0\ I]P(\tau) \tag{B2a}$$

in terms of the time-to-go $\tau=t_f-t$, initialized by

$$P(0)=\begin{bmatrix}p_1 I & 0\\ 0 & p_2 I\end{bmatrix} \tag{B2b}$$

The absence of a mean state penalty makes it doubly advantageous to propagate instead the inverse matrix

$$\Pi(\tau)=P(\tau)^{-1} \tag{B3}$$

The evolution equation for $\Pi(\tau)$ is obtained by pre- and post-multiplying Eqs. (B2) by $\Pi(\tau)$ and changing sign to yield

$$-\dot\Pi(\tau)=\begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix}\Pi(\tau)+\Pi(\tau)\begin{bmatrix}0 & 0\\ I & 0\end{bmatrix}-(1/r)\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix} \tag{B4a}$$

initialized by

$$\Pi(0)=\begin{bmatrix}(1/p_1)I & 0\\ 0 & (1/p_2)I\end{bmatrix} \tag{B4b}$$

The double advantage is that Eqs. (B4) are Lyapunov equations, hence *linear*, and moreover the terminal penalties p_i appear inverted, hence, may be allowed to diverge *before* solving the equation. [The latter would still be true even for $q_i=0$, although Eqs. (B4) would then become additional Riccati equations.] By partitioning

$$\Pi=\{\Pi_{ij}\},i,j=1,2,\ \text{with}\ \Pi_{12}=\Pi_{21}^T$$

Eqs. (B4) are reduced to the system

$$-\dot\Pi_{22}(\tau)=-(1/r)I,\qquad \Pi_{22}(0)=0 \tag{B5a}$$

$$-\dot\Pi_{12}(\tau)=\Pi_{22}(\tau),\qquad \Pi_{12}(0)=0 \tag{B5b}$$

$$-\dot\Pi_{11}(\tau)=2\Pi_{12}(\tau),\qquad \Pi_{11}(0)=0 \tag{B5c}$$

after letting $(1/p_i)\to 0$, $i=1,2$. Equations (B5) can be solved sequentially to yield

$$\Pi(\tau)=\frac{1}{r}\begin{bmatrix}⅓\tau^3 I & -½\tau^2 I\\ ½\tau^2 I & \tau I\end{bmatrix} \tag{B6}$$

which, by inversion, yields

$$P(\tau)=(2r/\tau^3)\begin{bmatrix}6I & 3\tau I\\ 3\tau I & 2\tau^2 I\end{bmatrix} \tag{B7}$$

(as stated without proof in Ref. 14).

Insertion of Eq. (B7) into Eq. (B1) yields the optimal feedback law

$$u(t) = -\{6/(t_f-t)^2\}x_1(t) - \{4/(t_f-t)\}x_2(t) \quad \text{(B8)}$$

which, when inserted into the double integrator equations (21), yields the time-varying linear vector equation

$$(t_f-t)^2\ddot{x}_1(t) + 4(t_f-t)\dot{x}_1(t) + 6x_1(t) = 0 \quad \text{(B9)}$$

(also stated without proof in Ref. 14), which has a regular singular point at $t=t_f$. However, Eq. (B9) is fortunately of "Euler type," with real roots for its indicial equation, leading to a simple closed-form solution of the form

$$x_1(t) = (t_f-t)^3 p + (t_f-t)^2 q \quad \text{(B10)}$$

whence

$$x_2(t) = -3(t_f-t)^2 p - 2(t_f-t)q \quad \text{(B11)}$$

and

$$u(t) = 6(t_f-t)p + 2q \quad \text{(B12)}$$

The vector coefficients p and q are obtained from Eqs. (B10) and (B11) with $t=0$, in terms of the initial data $\gamma(0)$ and $\dot{\gamma}(0) = \Gamma(\gamma(0))^{-1}\omega(0)$, as found in Ref. 14, to finally yield Eqs. (27-29).

Appendix C: Singularity Avoidance

Since one has

$$x_1(0)^T x_1(0) = \gamma(0)^T\gamma(0) < 1$$

by hypothesis, as well as $x_1(t_f) = 0$ by inspection of Eq. (27) or (B10), it follows that the constraint $\gamma_0(t) > 0$ will be violated during the maneuver [by virtue of Eq. (11)], if, and only if, it is violated at an interior local maximum of $x_1(t)^T x_1(t)$. Therefore, it is enough to examine the extrema of the normalized quadratic

$$3p^T p s^2 + 5p^T q s + 2q^T q = 0 \quad \text{(C1)}$$

for $0<s<1$, where $s = (t_f-t)/t_f$, as follows from examining the formula

$$\frac{d}{dt}\{x_1(t)^T x_1(t)\} = -(t_f-t)^3\{3(t_f-t)^2 p^T p + 5(t_f-t)p^T q + 2q^T q\} \quad \text{(C2)}$$

that is easily derived from Eq. (B10).

The feedback singularity will occur if, and only if, it occurs at $t_j = (1-s_j)t_f$ for $j=1,2$, where the s_j are the roots of the quadratic equation (C1). Hence it is enough to test the norm of $x_1(t)$ at two points at most, i.e.,

$$x_1(t_j)^T x_1(t_j) < 1, \qquad j=1,2 \quad \text{(C3)}$$

In particular, constraints of Eq. (C3) will be vacuously satisfied if the roots of Eq. (C1) lie outside the open interval (0,1), as is the case for rest-to-rest maneuvers (see Ref. 14). The constraints are also vacuously satisfied if the roots of Eq. (C1) are complex, which occurs if, and only if, the condition

$$\cos^2(p,q) < 24/25 \quad \text{(C4)}$$

is verified.

If the tests of Eqs. (C3) and (C4) fail, such as for very high initial tumbling rates, then the commands defined by Eq. (29) cannot be used. In this case, either a prior detumbling command can be used, as derived in Ref. 13, or one must choose a sufficiently large mean state penalty q_1 on x_1 in the performance index [Eq. (26*): In this case $P(\tau)$ of Eqs. (B1) and (B2) must be recalculated, and the locus of q_1 for which $\gamma_0(t)$ remains positive determined therefrom.

Still another solution is to use a minimum jerk, rather than an acceleration maneuver, which permits starting and ending with zero torques, and *restart* the optimal acceleration command at the actual state reached at the end of any midcourse saturation region, as done for robots in Ref. 19 and for spacecraft in Ref. 20.

Appendix D: Simulation Results

For the sake of comparison, the spacecraft with three identical orthogonal reaction wheels considered by Vadali and Junkins[7] was selected for simulation, as well as the same maneuver time and initial conditons, as given below. However, unlike Ref. 7, a least-mean-square attitude parameter acceleration maneuver, given by Eqs. (1) and (27-29), is used rather than a least-mean-square torque.

The spacecraft products of inertia were therefore chosen to be $I^0_{11} = 87.212$, $I^0_{22} = 86.067$, $I^0_{33} = 114.562$, $I^0_{ij} = -0.2237$ for $i \neq j$, and the common axial moments of inertia of the reaction wheels, all in kg·m² were $I^A_i = 0.05$, $i = 1,2,3$. (The presence of skew terms in I^0 in this example arises from a locked fourth skew wheel, used for backup in the NASA standard configuration; cf. Ref. 12.)

The initial Euler parameters were chosen, as in Ref. 7, to correspond to a virtual rotation of $\phi = 100$ deg about the

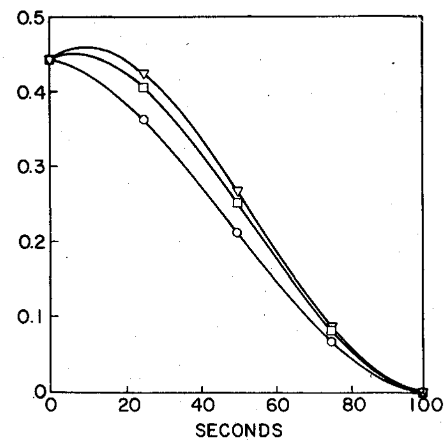

Fig. D1 Attitude parameters γ_i □, $i=1$; ∇, $i=2$; ○, $i=3$.

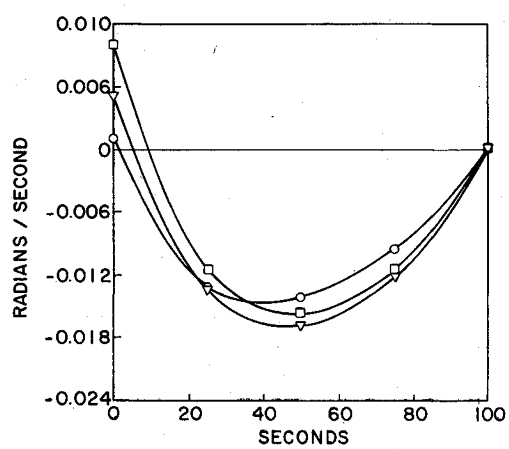

Fig. D2 Angular velocities ω_i in rad/s; □, $i=1$; ∇, $i=2$; ○, $i=3$.

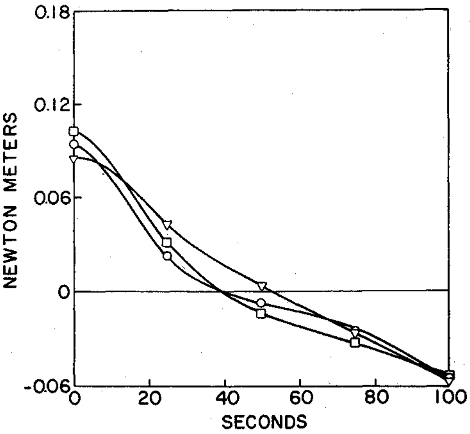

Fig. D3 Torques τ_i in N/m; □, $i=1$; ▽, $i=2$; ○, i=3.

axis $e=(1/\sqrt{3})$ col (1,1,1), to yield

$$\gamma_i(0)=\beta_i(0)=0.44227597, \quad i=1,2,3$$

$$\gamma_0(0)=\beta_0(0)=0.64278761$$

Again following Ref. 7, the initial angular velocity components in body axes in radians per second were chosen to be

$$\omega_1(0)=0.01, \quad \omega_2(0)=0.005, \quad \omega_3(0)=0.001$$

The reaction wheels were supposed to be initially at rest,

$$\Omega_i(0)=0.0, \quad i=1,2,3$$

It follows from Eqs. (13) and (14) that the inertial components of the total angular momentum in kg·m^2 rad/s become

$$h_1^I=0.22030, \quad h_2^I=1.90914, \quad h_3^I=0.28042$$

It also follows from Eqs. (16a) and (17) that

$$\dot{\gamma}_1(0)=0.0023294, \ \dot{\gamma}_2(0)=0.0035972, \ \dot{\gamma}_3(0)=-0.0007843$$

The final time was also chosen, as in Ref. 7, to be $t_f=$ 100.0 s.

Figure D1 shows the simulated evolution of the Euler parameters $\gamma_i(t)$, which are seen to have been driven exactly to their zero targets at the end of the maneuver, as expected. Figure D2 shows the simulated evolution of the body components $\omega_i(t)$ of angular velocity in radians per second, which are also seen to have been driven exactly to their targets. Figure D3 shows the simulated evolution of the torque components $\tau_i(t)$, which are seen to have remained bounded throughout the maneuver.

It should be noted finally that the axial angular velocity histories $\Omega_i(t)$ of the reaction wheels can be synthesized if desired from the torque component histories $\tau_i(t)$ and the spacecraft angular velocity component histories $\omega_i(t)$ by integration of Eq. (3), which yields

$$\Omega_i(t)=\Omega_i(0)+\omega_i(0)-\omega_i(t)+(1/I_i^A)\int_0^t \tau_i(s)\mathrm{d}s \qquad \text{(D1)}$$

for $i=1,2,3$ and $0\le t\le t_f$. Since, as can be seen from Fig. D3, the torque components cross over from positive to negative approximately halfway during the terminal control maneuver, the integrals in Eq. (D1) can be expected to be small for $t=t_f$. Moreover, since $\omega_i(t_f)=0$ and $\Omega_i(0)=0$ by design, it follows that the *final* reaction wheel angular velocities $\Omega_i(t_f)$ should be comparable to the *initial* spacecraft angular velocity components $\omega_i(0)$ (with more integrated torque contributions the smaller the wheel inertias I_i^A). This behavior is consistent with the "momentum transfer" character of the maneuver.

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